Bodové symetrie rovnice vedení tepla

- nyní už stručněji s využitím Mathematicy (viz notebook na webu)
  - uvažujme rovnici $u_{xx} = u_t$

    a hledejme $X = \xi^x(x,t,u)\frac{\partial}{\partial x} + \xi^t(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}$
  - z podmínky $X^{(2)}(u_{xx}-u_t)\big|_{u_{xx}=u_t} = 0$ (*)

    dostaneme (mimo jiné) podmínky

$$\frac{\partial \xi^x}{\partial u} = 0 = \frac{\partial \xi^t}{\partial u} = \frac{\partial \xi^t}{\partial x} = \frac{\partial^2 \eta}{\partial u^2} \quad \text{a} \quad 2\frac{\partial \xi^x}{\partial x} = \frac{\partial \xi^t}{\partial t}$$

z nichž určíme 2. ansatz

$$\xi^t(x,t,u) = \tau(t), \quad \xi^x(x,t,u) = \tfrac{1}{2}\tau'(t)x + \chi(t), \quad \eta(x,t,u) = \alpha(x,t)u + \beta(x,t)$$

- opětovným dosazením do (*) vyjde

$$\frac{\partial \alpha}{\partial x} = -\frac{1}{4}\tau''(t)x - \frac{1}{2}\chi'(t) \implies \alpha \text{ je kvadratické v } x$$

a tedy 3. ansatz je jako v 2. ansatzu, jen

$$\alpha(x,t) = -\frac{1}{8}\tau''(t)x^2 - \frac{1}{2}\chi'(t)x + \gamma(t)$$

- konečně máme podmínky

$$\chi''(t) = 0, \quad \tau'''(t) = 0 \quad \text{a} \quad 4\gamma'(t) = -\tau''(t)$$

$\implies$ χ je lineární v t, τ kvadratická v t, a γ lineární v t

- odtud výsledek (6 nezávislých parametrů)

$$\xi^x(x,t,u) = c_1 + c_4 x + 2c_5 t + 4c_6 xt$$
$$\xi^t(x,t,u) = c_2 + 2c_4 t + 4c_6 t^2$$
$$\eta(x,t,u) = (c_3 - c_5 x - 2c_6 t - c_6 x^2)u + \beta(x,t)$$

kde $\beta(x,t)$ je libovolné řešení rovnice vedení tepla $\beta_{xx} = \beta_t$
(toto je důsledek linearity rovnice)

tj. máme 6 lin. nezávislých netriviálních gen.

$X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial t}$ $\iff$ translace v x: $\tilde{x} = x+\varepsilon$, $\tilde{t} = t$, $\tilde{u} = u$, translace v t: $\tilde{t} = t+\varepsilon$, $\tilde{x} = x$, $\tilde{u} = u$

$X_3 = u\frac{\partial}{\partial u}$, $X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$ $\iff$ škálování v u: $\tilde{u} = e^{\varepsilon}u$, $\tilde{x} = x$, $\tilde{t} = t$, škálování t a x: $\tilde{x} = e^{\varepsilon}x$, $\tilde{t} = e^{2\varepsilon}t$, $\tilde{u} = u$

$X_5 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}$ $\iff$ Galileova transf. $\tilde{x} = x + 2\varepsilon t$, $\tilde{t} = t$, $\tilde{u} = u e^{-x\varepsilon - \varepsilon^2 t}$

$X_6 = 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (2t + x^2)u\frac{\partial}{\partial u}$ $\iff$ „projektivní" transf. v t

$$\tilde{x} = \frac{x}{1-4t\varepsilon}, \quad \tilde{t} = \frac{t}{1-4t\varepsilon}$$
$$\tilde{u} = u\sqrt{1-4\varepsilon t}\; e^{-\frac{\varepsilon x^2}{1-4t\varepsilon}}$$

a ∞-rozm. Lieovu algebru ve tvaru

$X_{\infty} = \beta(x,t)\frac{\partial}{\partial u}$ $\iff$ $\tilde{x} = x$, $\tilde{t} = t$, $\tilde{u} = u + \varepsilon\beta(x,t)$

Bodové symetrie 1D Schrödingerovy rovnice pro volnou částici

- téměř totožné s rovnicí vedení tepla, jen komplexní
- hledejme bod. symetrie rce

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} \Rightarrow i\frac{\partial \psi}{\partial t} = -\lambda \frac{\partial^2 \psi}{\partial x^2}, \text{ kde } \lambda = \frac{\hbar}{2m}$$

generované $X = \xi^x(x,t,\psi)\frac{\partial}{\partial x} + \xi^t(x,t,\psi)\frac{\partial}{\partial t} + \eta(x,t,\psi)\frac{\partial}{\partial \psi}$ 1. ansatz

- z podmínky

$$X^{(2)}\left(i\frac{\partial \psi}{\partial t} + \lambda\frac{\partial^2 \psi}{\partial x^2}\right)\Bigg|_{\frac{\partial \psi}{\partial t} = +i\lambda\frac{\partial^2 \psi}{\partial x^2}} = 0 \qquad (*)$$

dostaneme (mimo jiné)

$$\left.\begin{aligned} &\frac{\partial \xi^t}{\partial \psi} = 0, \quad \frac{\partial \xi^t}{\partial x} = 0 \Rightarrow \xi^t = \xi^t(t) = \tau(t) \\ \text{a dále } &\frac{\partial \xi^x}{\partial \psi} = -i\lambda \frac{\partial^2 \xi^t}{\partial x \partial \psi} = 0 \Rightarrow \xi^x = \xi^x(x,t) \\ &\frac{\partial^2 \eta}{\partial \psi^2} = 2\frac{\partial^2 \xi^x}{\partial x \partial \psi} = 0 \Rightarrow \eta = \eta^\psi(x,t)\,\psi + \eta^0(x,t) \end{aligned}\right\} \text{2. ansatz}$$

- dosazení opět do $(*)$:

$$\frac{d\tau}{dt} = 2\frac{\partial \xi^x(x,t)}{\partial x} \Rightarrow \text{linearita } \xi^x \text{ v } x \Rightarrow \xi^x = \frac{1}{2}\frac{d\tau}{dt}x + \sigma(t)$$

$$\xi^t = \tau(t)$$

$$\underbrace{-i\frac{\partial \xi^x}{\partial t}}_{\text{lineární v } x} + 2\lambda\frac{\partial \eta^\psi}{\partial x} = \lambda\frac{\partial^2 \xi^x}{\partial x^2} = 0 \Rightarrow \eta^\psi \text{ kvadratické v } x$$

$$\underbrace{\eta = \left(\alpha(t)x^2 + \beta(t)x + \gamma(t)\right)\psi + \eta^0(x,t)}_{\text{3. ansatz}}$$

- opětovné dosazení do $(*)$:

$$\frac{d\alpha}{dt} = 0, \quad \frac{d\beta}{dt} = 0 \qquad \text{konstanty}$$

$$2\lambda\alpha + i\frac{d\gamma}{dt} = 0, \quad 2\lambda\beta - i\frac{d\sigma}{dt} = 0 \Rightarrow \gamma \text{ a } \sigma \text{ lineární v } t$$

$$8\lambda\alpha = i\frac{d^2\tau}{dt^2} \Rightarrow \tau \text{ kvadratické v } t$$

$\Rightarrow$ 4. ansatz

$$\boxed{\begin{aligned} \xi^x &= c_1 + c_3 x + c_5 t + c_6 t x \\ \xi^t &= c_2 + 2c_3 t + c_6 t^2 \\ \eta &= \left(c_4 + \frac{ic_5 x}{2\lambda} + \frac{ic_6 x^2}{4\lambda} - \frac{c_6 t}{2}\right)\psi + \eta^0(x,t) \end{aligned}}$$

kde $\eta^0(x,t)$ je lib. řešení Schr. rovnice

• konečné transformace pro jediné $c_i = 1$, ostatní nulové

$X_1 = \frac{\partial}{\partial x}$ ... translace v x: $\quad \tilde{x} = x + \varepsilon$, $\tilde{t} = t$, $\tilde{\varphi} = \varphi$

$X_2 = \frac{\partial}{\partial t}$ ... translace v t: $\quad \tilde{x} = x$, $\tilde{t} = t + \varepsilon$, $\tilde{\varphi} = \varphi$

$X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$ ... škálování: $\quad \tilde{x} = xe^{\varepsilon}$, $\tilde{t} = te^{2\varepsilon}$, $\tilde{\varphi} = \varphi$

$X_4 = c\varphi\frac{\partial}{\partial \varphi}$ ... škálování a fáze, c obecně komplexní

pro $c = 1$: $\quad \tilde{x} = x$, $\tilde{t} = t$, $\tilde{\varphi} = e^{\varepsilon}\varphi$

pro $c = i$: $\quad \tilde{x} = x$, $\tilde{t} = t$, $\tilde{\varphi} = e^{i\varepsilon}\varphi$

$X_5 = t\frac{\partial}{\partial x} + \frac{ix}{2\lambda}\varphi\frac{\partial}{\partial \varphi}$ ... Galileova transf. se změnou fáze φ

(pro $\varepsilon = v$) $\quad \tilde{x} = x + vt$, $\tilde{t} = t$, $\tilde{\varphi} = \varphi\, e^{i(2vx + tv^2)/4\lambda} = \varphi\, e^{i(mvx + \frac{1}{2}mv^2 t)/\hbar}$

ovšem pozor, řešení $\varphi = \Theta(x,t)$ se ztransformuje na jiné řešení dané implicitně $\tilde{\varphi} = G(F(\tilde{x},\tilde{t},\tilde{\varphi},\varepsilon^{-1}), \Theta(F(\tilde{x},\tilde{t},\tilde{\varphi},\varepsilon^{-1}),\varepsilon)$

a tedy např. z $\Theta(x,t) = A$ bude

$$\tilde{\varphi} = A\, e^{\frac{i}{\hbar}\left[mv(\tilde{x} - v\tilde{t}) + \frac{1}{2}mv^2\tilde{t}\right]} \Rightarrow$$

nové řešení (odulnkované) $\quad \varphi(x,t) = A\, e^{\frac{i}{\hbar}\left(mvx - \frac{1}{2}mv^2 t\right)}$ rovinná vlna

$X_6 = tx\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + \left(\frac{ix^2}{4\lambda} - \frac{t}{2}\right)\varphi\frac{\partial}{\partial \varphi}$ .. projektivní transformace

$$\tilde{t} = \frac{t}{1 - \varepsilon t}, \quad \tilde{x} = \frac{x}{1 - \varepsilon t}, \quad \tilde{\varphi} = \varphi\sqrt{1 - \varepsilon t}\, e^{\frac{im\varepsilon x^2}{2\hbar(1 - \varepsilon t)}}$$

opět bychom z konstantního řešení $\Theta(x,t) = A$ dostali

netriviální řešení $\quad \varphi(x,t) = \frac{A}{\sqrt{1 + \varepsilon t}}\, e^{\frac{im\varepsilon x^2}{2\hbar(1 + \varepsilon t)}}$

• totéž bychom dostali pro $\varphi = \varphi_R + i\varphi_I \Rightarrow$

$$\Rightarrow \frac{\partial \varphi_I}{\partial t} = \lambda\frac{\partial^2 \varphi_R}{\partial x^2}$$

$$\frac{\partial \varphi_R}{\partial t} = -\lambda\frac{\partial^2 \varphi_I}{\partial x^2}$$

jen nyní zvlášť škálování $X_4 = \varphi_R\frac{\partial}{\partial \varphi_R} + \varphi_I\frac{\partial}{\partial \varphi_I}$

a násobení fází $X_4' = -\varphi_R\frac{\partial}{\partial \varphi_I} + \varphi_I\frac{\partial}{\partial \varphi_R}$

(rotace v (φ_R, φ_I))

a nyní

$$X_5 = t\frac{\partial}{\partial x} + \frac{x}{2\lambda}\left(\varphi_R\frac{\partial}{\partial \varphi_i} - \varphi_I\frac{\partial}{\partial \varphi_R}\right)$$

$$X_6 = tx\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - \frac{t}{2}\left(\varphi_R\frac{\partial}{\partial \varphi_R} + \varphi_I\frac{\partial}{\partial \varphi_I}\right) + \frac{x^2}{4\lambda}\left(\varphi_R\frac{\partial}{\partial \varphi_i} - \varphi_I\frac{\partial}{\partial \varphi_R}\right)$$