Point symmetries of heat equation

Run TMF064.Package.m first!

```
Clear["Global`*"]
```

Variables and differential equations in the form R(x,u,∂u,...) = 0

```
(* Independent variables *)
IndepVar = {x, t};
(* Dependent variables *)
DepVar = {u};
(* PDEs, only the functions R(...) without "== 0" *)
PDEs = {D[u[x, t], t] - D[u[x, t], x, x]}
```

```
{u^(0,1)[x, t] - u^(2,0)[x, t]}
```

Expression to substitute for in the infinitesimal criterion of invariance, when dealing with the heat equation all time derivatives can be replaced by space derivatives

```
subs = {D[u[x, t], t]};
sol = Solve[PDEs == 0, subs]
```

```
{{u^(0,1)[x, t] → u^(2,0)[x, t]}}
```

Finding point symmetries by using a more and more specific ansatz

General ansatz

```
(* Infinitesimals for all variables *)
ξ[t] = Ξt[x, t, u[x, t]];
ξ[x] = Ξx[x, t, u[x, t]];
η[u] = H[x, t, u[x, t]];
(* Next expression should return zeroes
 if infinitesimals give a point symmetry of PDEs *)
zero = CheckPointSymmetryOfDE[PDEs, subs, IndepVar, DepVar, ξ, η]
```

```
u[0] = u[x, t]
u[1] = u^(0,1)[x, t]
u[2] = u^(1,0)[x, t]
u[3] = u^(0,2)[x, t]
u[4] = u^(1,1)[x, t]
u[5] = u^(2,0)[x, t]
```

```
{2 u[2] u[4] Ξt^(0,0,1)[x, t, u[0]] + 2 u[2] u[5] Ξx^(0,0,1)[x, t, u[0]] -
  u[2]^2 H^(0,0,2)[x, t, u[0]] + u[2]^2 u[5] Ξt^(0,0,2)[x, t, u[0]] + u[2]^3 Ξx^(0,0,2)[x, t, u[0]] +
  H^(0,1,0)[x, t, u[0]] - u[5] Ξt^(0,1,0)[x, t, u[0]] - u[2] Ξx^(0,1,0)[x, t, u[0]] +
  2 u[4] Ξt^(1,0,0)[x, t, u[0]] + 2 u[5] Ξx^(1,0,0)[x, t, u[0]] - 2 u[2] H^(1,0,1)[x, t, u[0]] +
  2 u[2] u[5] Ξt^(1,0,1)[x, t, u[0]] + 2 u[2]^2 Ξx^(1,0,1)[x, t, u[0]] -
  H^(2,0,0)[x, t, u[0]] + u[5] Ξt^(2,0,0)[x, t, u[0]] + u[2] Ξx^(2,0,0)[x, t, u[0]]}
```

```
(* Coefficients of the polynomial in u[...] should be zero *)
Column[GetConditionsForPointSymmetries[zero, Table[u[j], {j, 1, 5}]]]
```

```
2 Ξt^(0,0,1)[x, t, u[0]]
Ξt^(0,0,2)[x, t, u[0]]
Ξx^(0,0,2)[x, t, u[0]]
2 Ξt^(1,0,0)[x, t, u[0]]
2 (Ξx^(0,0,1)[x, t, u[0]] + Ξt^(1,0,1)[x, t, u[0]])
-H^(0,0,2)[x, t, u[0]] + 2 Ξx^(1,0,1)[x, t, u[0]]
H^(0,1,0)[x, t, u[0]] - H^(2,0,0)[x, t, u[0]]
-Ξt^(0,1,0)[x, t, u[0]] + 2 Ξx^(1,0,0)[x, t, u[0]] + Ξt^(2,0,0)[x, t, u[0]]
-Ξx^(0,1,0)[x, t, u[0]] - 2 H^(1,0,1)[x, t, u[0]] + Ξx^(2,0,0)[x, t, u[0]]
```

Second ansatz

```
ξ[t] = τ[t];
ξ[x] = 1/2 D[τ[t], t] x + χ[t];
η[u] = α[x, t] u[x, t] + β[x, t];
zero = CheckPointSymmetryOfDE[PDEs, subs, IndepVar, DepVar, ξ, η]
```

```
u[0] = u[x, t]
u[1] = u^(0,1)[x, t]
u[2] = u^(1,0)[x, t]
u[3] = u^(0,2)[x, t]
u[4] = u^(1,1)[x, t]
u[5] = u^(2,0)[x, t]
```

$$\left\{-u[2]\left(\chi'[t]+\frac{1}{2}\,x\,\tau''[t]\right)+u[0]\,\alpha^{(0,1)}[x,t]+\beta^{(0,1)}[x,t]-2\,u[2]\,\alpha^{(1,0)}[x,t]-u[0]\,\alpha^{(2,0)}[x,t]-\beta^{(2,0)}[x,t]\right\}$$

```
Column[GetConditionsForPointSymmetries[zero, Table[u[j], {j, 0, 5}]]]
```

$-2\chi'[t] - x\,\tau''[t] - 4\,\alpha^{(1,0)}[x, t]$

$2\left(\alpha^{(0,1)}[x, t] - \alpha^{(2,0)}[x, t]\right)$

$2\left(\beta^{(0,1)}[x, t] - \beta^{(2,0)}[x, t]\right)$

Third ansatz

```
ξ[t] = τ[t];
ξ[x] = 1/2 D[τ[t], t] x + χ[t];
η[u] = (-1/8 D[τ[t], t, t] x^2 - 1/2 D[χ[t], t] x + γ[t]) u[x, t] + β[x, t];
zero = CheckPointSymmetryOfDE[PDEs, subs, IndepVar, DepVar, ξ, η]
```

$u[0] = u[x, t]$

$u[1] = u^{(0,1)}[x, t]$

$u[2] = u^{(1,0)}[x, t]$

$u[3] = u^{(0,2)}[x, t]$

$u[4] = u^{(1,1)}[x, t]$

$u[5] = u^{(2,0)}[x, t]$

$$\left\{u[0]\,\gamma'[t] + \frac{1}{4}u[0]\,\tau''[t] - \frac{1}{2}x\,u[0]\,\chi''[t] - \frac{1}{8}x^2\,u[0]\,\tau^{(3)}[t] + \beta^{(0,1)}[x, t] - \beta^{(2,0)}[x, t]\right\}$$

```
Column[GetConditionsForPointSymmetries[zero, Flatten[{Table[u[j], {j, 0, 5}], x}] ]]
```

$2\left(4\,\gamma'[t] + \tau''[t]\right)$

$-4\,\chi''[t]$

$-\tau^{(3)}[t]$

$8\left(\beta^{(0,1)}[x, t] - \beta^{(2,0)}[x, t]\right)$

The last ansatz

```
ξ[x] = c[1] + c[4] x + 2 c[5] t + 4 c[6] x t;
ξ[t] = c[2] + 2 c[4] t + 4 c[6] t^2;
η[u] = (c[3] - c[5] x - 2 c[6] t - c[6] x^2) u[x, t] + β[x, t];
zero = CheckPointSymmetryOfDE[PDEs, subs, IndepVar, DepVar, ξ, η]
```

$u[0] = u[x, t]$

$u[1] = u^{(0,1)}[x, t]$

$u[2] = u^{(1,0)}[x, t]$

$u[3] = u^{(0,2)}[x, t]$

$u[4] = u^{(1,1)}[x, t]$

$u[5] = u^{(2,0)}[x, t]$

$$\left\{\beta^{(0,1)}[x, t] - \beta^{(2,0)}[x, t]\right\}$$

Infinitesimal generators, point transformations and commutator table from the last ansatz

```
ShowPointSymmetriesAndCommutationRelations[
  X, f, ∈, IndepVar, DepVar, ξ, η, c, 6, {β[x, t] → 0}]
```

Infinitesimal operators:

$X[1]f[x, t, u] = f^{(1,0,0)}[x, t, u]$

$X[2]f[x, t, u] = f^{(0,1,0)}[x, t, u]$

$X[3]f[x, t, u] = u\, f^{(0,0,1)}[x, t, u]$

$X[4]f[x, t, u] = 2\, t\, f^{(0,1,0)}[x, t, u] + x\, f^{(1,0,0)}[x, t, u]$

$X[5]f[x, t, u] = -u\, x\, f^{(0,0,1)}[x, t, u] + 2\, t\, f^{(1,0,0)}[x, t, u]$

$X[6]f[x, t, u] = u\left(-2\, t - x^2\right) f^{(0,0,1)}[x, t, u] + 4\, t^2 f^{(0,1,0)}[x, t, u] + 4\, t\, x\, f^{(1,0,0)}[x, t, u]$

Solve: Inconsistent or redundant transcendental equation. After reduction, the bad equation is $-x^2 + 2\, t\, \text{InverseFunction}[\#1\, e^{\#1}\ \&, 1, 1]\left[-\frac{e^{-\frac{C[\ll1\gg]^2}{2\, C[\ll1\gg]}}\, C[2]^2}{2\, C[1]}\right] == 0.$

Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information.

Corresponding global transformations:

X[1] gives $\{x[\epsilon] \to x + \epsilon,\ t[\epsilon] \to t,\ u[\epsilon] \to u\}$

X[2] gives $\{x[\epsilon] \to x,\ t[\epsilon] \to t + \epsilon,\ u[\epsilon] \to u\}$

X[3] gives $\{x[\epsilon] \to x,\ t[\epsilon] \to t,\ u[\epsilon] \to u\, e^{\epsilon}\}$

X[4] gives $\{x[\epsilon] \to x\, e^{\epsilon},\ t[\epsilon] \to t\, e^{2\epsilon},\ u[\epsilon] \to u\}$

X[5] gives $\{t[\epsilon] \to t,\ x[\epsilon] \to x + 2\, t\, \epsilon,\ u[\epsilon] \to u\, e^{-\epsilon\,(x + t\, \epsilon)}\}$

X[6] gives $\left\{t[\epsilon] \to \frac{t}{1 - 4\, t\, \epsilon},\ x[\epsilon] \to \frac{x}{1 - 4\, t\, \epsilon},\ u[\epsilon] \to u\, e^{-\frac{x^2}{4\, t - 16\, t^2\, \epsilon}} \sqrt{-t\, e^{\frac{x^2}{2\, t}}} \sqrt{-\frac{1}{t} + 4\, \epsilon}\right\}$

Commutator table:

	1	2	3	4	5	6
1	0	0	0	X[1]	-X[3]	2 X[5]
2	0	0	0	2 X[2]	2 X[1]	-2 X[3] + 4 X[4]
3	0	0	0	0	0	0
4	-X[1]	-2 X[2]	0	0	X[5]	2 X[6]
5	X[3]	-2 X[1]	0	-X[5]	0	0
6	-2 X[5]	2 X[3] - 4 X[4]	0	-2 X[6]	0	0