Poincare group of symmetry leads to Klein-Gordon equation

Run TMF064.Package.m first!

```
Clear["Global`*"]
```

- **Is the Klein-Gordon equation the only linear scalar equation of the second order which is invariant under the Poincare group of transformations?**

```
(* Independent variables *)
IndepVar = {x[0], x[1], x[2], x[3]};
ivar = Delete[IndepVar, 0];
(* Dependent variables *)
DepVar = {ψ};
(* PDE *)
PDEs = {w ψ[ivar] + Sum[a[k] D[ψ[ivar], x[k]], {k, 0, 3}] +
   Sum[Sum[If[k > l, 0, b[k, l] D[ψ[ivar], x[k], x[l]]], {l, 0, 3}], {k, 0, 3}]}
```

```
{w ψ[x[0], x[1], x[2], x[3]] +
  a[3] ψ^(0,0,0,1)[x[0], x[1], x[2], x[3]] + b[3, 3] ψ^(0,0,0,2)[x[0], x[1], x[2], x[3]] +
  a[2] ψ^(0,0,1,0)[x[0], x[1], x[2], x[3]] + b[2, 3] ψ^(0,0,1,1)[x[0], x[1], x[2], x[3]] +
  b[2, 2] ψ^(0,0,2,0)[x[0], x[1], x[2], x[3]] + a[1] ψ^(0,1,0,0)[x[0], x[1], x[2], x[3]] +
  b[1, 3] ψ^(0,1,0,1)[x[0], x[1], x[2], x[3]] + b[1, 2] ψ^(0,1,1,0)[x[0], x[1], x[2], x[3]] +
  b[1, 1] ψ^(0,2,0,0)[x[0], x[1], x[2], x[3]] + a[0] ψ^(1,0,0,0)[x[0], x[1], x[2], x[3]] +
  b[0, 3] ψ^(1,0,0,1)[x[0], x[1], x[2], x[3]] + b[0, 2] ψ^(1,0,1,0)[x[0], x[1], x[2], x[3]] +
  b[0, 1] ψ^(1,1,0,0)[x[0], x[1], x[2], x[3]] + b[0, 0] ψ^(2,0,0,0)[x[0], x[1], x[2], x[3]]}
```

Expression to substitute for in the infinitesimal criterion of invariance

```
subs = {ψ[ivar]};
sol = Solve[PDEs == 0, subs]
```

```
{{ψ[x[0], x[1], x[2], x[3]] →
   1/w (-a[3] ψ^(0,0,0,1)[x[0], x[1], x[2], x[3]] - b[3, 3] ψ^(0,0,0,2)[x[0],
        x[1], x[2], x[3]] - a[2] ψ^(0,0,1,0)[x[0], x[1], x[2], x[3]] - b[2, 3]
        ψ^(0,0,1,1)[x[0], x[1], x[2], x[3]] - b[2, 2] ψ^(0,0,2,0)[x[0], x[1], x[2], x[3]] -
      a[1] ψ^(0,1,0,0)[x[0], x[1], x[2], x[3]] - b[1, 3]
        ψ^(0,1,0,1)[x[0], x[1], x[2], x[3]] - b[1, 2] ψ^(0,1,1,0)[x[0], x[1], x[2], x[3]] -
      b[1, 1] ψ^(0,2,0,0)[x[0], x[1], x[2], x[3]] - a[0]
        ψ^(1,0,0,0)[x[0], x[1], x[2], x[3]] - b[0, 3] ψ^(1,0,0,1)[x[0], x[1], x[2], x[3]] -
      b[0, 2] ψ^(1,0,1,0)[x[0], x[1], x[2], x[3]] - b[0, 1]
        ψ^(1,1,0,0)[x[0], x[1], x[2], x[3]] - b[0, 0] ψ^(2,0,0,0)[x[0], x[1], x[2], x[3]])}}
```

Infinitesimals of the Poincare group and infinitesimal criterion of invariance

```
nc = 10;
ξ[x[0]] = c[7] + c[4] x[1] + c[5] x[2] + c[6] x[3];
ξ[x[1]] = c[8] + c[2] x[3] - c[3] x[2] + c[4] x[0];
ξ[x[2]] = c[9] - c[1] x[3] + c[3] x[1] + c[5] x[0];
ξ[x[3]] = c[10] + c[1] x[2] - c[2] x[1] + c[6] x[0]; η[ψ] = 0;
zero = CheckPointSymmetryOfDE[PDEs, subs, IndepVar, DepVar, ξ, η]
```

ψ[0] = ψ[x[0], x[1], x[2], x[3]]

ψ[1] = ψ^(1,0,0,0)[x[0], x[1], x[2], x[3]]

ψ[2] = ψ^(0,1,0,0)[x[0], x[1], x[2], x[3]]

ψ[3] = ψ^(0,0,1,0)[x[0], x[1], x[2], x[3]]

ψ[4] = ψ^(0,0,0,1)[x[0], x[1], x[2], x[3]]

ψ[5] = ψ^(2,0,0,0)[x[0], x[1], x[2], x[3]]

ψ[6] = ψ^(1,1,0,0)[x[0], x[1], x[2], x[3]]

ψ[7] = ψ^(1,0,1,0)[x[0], x[1], x[2], x[3]]

ψ[8] = ψ^(1,0,0,1)[x[0], x[1], x[2], x[3]]

ψ[9] = ψ^(0,2,0,0)[x[0], x[1], x[2], x[3]]

ψ[10] = ψ^(0,1,1,0)[x[0], x[1], x[2], x[3]]

ψ[11] = ψ^(0,1,0,1)[x[0], x[1], x[2], x[3]]

ψ[12] = ψ^(0,0,2,0)[x[0], x[1], x[2], x[3]]

ψ[13] = ψ^(0,0,1,1)[x[0], x[1], x[2], x[3]]

ψ[14] = ψ^(0,0,0,2)[x[0], x[1], x[2], x[3]]

```
{-a[3] (c[6] ψ[1] + c[2] ψ[2] - c[1] ψ[3]) -
  a[2] (c[5] ψ[1] - c[3] ψ[2] + c[1] ψ[4]) - a[1] (c[4] ψ[1] + c[3] ψ[3] - c[2] ψ[4]) -
  a[0] (c[4] ψ[2] + c[5] ψ[3] + c[6] ψ[4]) - 2 b[0, 0] (c[4] ψ[6] + c[5] ψ[7] + c[6] ψ[8]) -
  2 b[1, 1] (c[4] ψ[6] + c[3] ψ[10] - c[2] ψ[11]) -
  b[0, 1] (c[3] ψ[7] - c[2] ψ[8] + c[4] (ψ[5] + ψ[9]) + c[5] ψ[10] + c[6] ψ[11]) -
  2 b[3, 3] (c[6] ψ[8] + c[2] ψ[11] - c[1] ψ[13]) -
  2 b[2, 2] (c[5] ψ[7] - c[3] ψ[10] + c[1] ψ[13]) -
  b[1, 2] (c[5] ψ[6] + c[4] ψ[7] - c[3] ψ[9] + c[1] ψ[11] + c[3] ψ[12] - c[2] ψ[13]) -
  b[0, 2] (-c[3] ψ[6] + c[1] ψ[8] + c[4] ψ[10] + c[5] (ψ[5] + ψ[12]) + c[6] ψ[13]) -
  b[2, 3] (c[6] ψ[7] + c[5] ψ[8] + c[2] ψ[10] - c[3] ψ[11] - c[1] ψ[12] + c[1] ψ[14]) -
  b[1, 3] (c[6] ψ[6] + c[4] ψ[8] + c[2] ψ[9] - c[1] ψ[10] + c[3] ψ[13] - c[2] ψ[14]) -
  b[0, 3] (c[2] ψ[6] - c[1] ψ[7] + c[4] ψ[11] + c[5] ψ[13] + c[6] (ψ[5] + ψ[14]))}
```

If it is not zero then find the equations for coefficients

```
variables = Flatten[Table[ψ[j], {j, 0, 14}]] ;
variables = Flatten[Union[variables, Table[c[j], {j, 1, nc}]]]
Column[GetConditionsForPointSymmetries[zero, variables]]
```

```
{c[1], c[2], c[3], c[4], c[5], c[6], c[7], c[8], c[9], c[10], ψ[0], ψ[1], ψ[2],
 ψ[3], ψ[4], ψ[5], ψ[6], ψ[7], ψ[8], ψ[9], ψ[10], ψ[11], ψ[12], ψ[13], ψ[14]}
```

```
-a[0]
-a[1]
a[1]
-a[2]
a[2]
-a[3]
a[3]
-b[0, 1]
b[0, 1]
-b[0, 2]
b[0, 2]
-b[0, 3]
b[0, 3]
-2 (b[0, 0] + b[1, 1])
-b[1, 2]
b[1, 2]
-b[1, 3]
b[1, 3]
-2 (b[1, 1] - b[2, 2])
-2 (b[0, 0] + b[2, 2])
-b[2, 3]
b[2, 3]
2 (b[1, 1] - b[3, 3])
-2 (b[2, 2] - b[3, 3])
-2 (b[0, 0] + b[3, 3])
```

Infinitesimal generators, point transformations and commutator table from the last ansatz

```
ShowPointSymmetriesAndCommutationRelations[X, f, ϵ, IndepVar, DepVar, ξ, η, c, 10, {}]
```

Infinitesimal operators:

X[1]f[x[0], x[1], x[2], x[3], ψ] =
$x[2]\, f^{(0,0,0,1,0)}[x[0], x[1], x[2], x[3], \psi] - x[3]\, f^{(0,0,1,0,0)}[x[0], x[1], x[2], x[3], \psi]$

X[2]f[x[0], x[1], x[2], x[3], ψ] =
$-x[1]\, f^{(0,0,0,1,0)}[x[0], x[1], x[2], x[3], \psi] + x[3]\, f^{(0,1,0,0,0)}[x[0], x[1], x[2], x[3], \psi]$

X[3]f[x[0], x[1], x[2], x[3], ψ] =
$x[1]\, f^{(0,0,1,0,0)}[x[0], x[1], x[2], x[3], \psi] - x[2]\, f^{(0,1,0,0,0)}[x[0], x[1], x[2], x[3], \psi]$

X[4]f[x[0], x[1], x[2], x[3], ψ] =
$x[0]\, f^{(0,1,0,0,0)}[x[0], x[1], x[2], x[3], \psi] + x[1]\, f^{(1,0,0,0,0)}[x[0], x[1], x[2], x[3], \psi]$

X[5]f[x[0], x[1], x[2], x[3], ψ] =
$x[0]\, f^{(0,0,1,0,0)}[x[0], x[1], x[2], x[3], \psi] + x[2]\, f^{(1,0,0,0,0)}[x[0], x[1], x[2], x[3], \psi]$

X[6]f[x[0], x[1], x[2], x[3], ψ] =
$x[0]\, f^{(0,0,0,1,0)}[x[0], x[1], x[2], x[3], \psi] + x[3]\, f^{(1,0,0,0,0)}[x[0], x[1], x[2], x[3], \psi]$

X[7]f[x[0], x[1], x[2], x[3], ψ] = $f^{(1,0,0,0,0)}[x[0], x[1], x[2], x[3], \psi]$

X[8]f[x[0], x[1], x[2], x[3], ψ] = $f^{(0,1,0,0,0)}[x[0], x[1], x[2], x[3], \psi]$

$X[9]f[x[0], x[1], x[2], x[3], \psi] = f^{(0,0,1,0,0)}[x[0], x[1], x[2], x[3], \psi]$

$X[10]f[x[0], x[1], x[2], x[3], \psi] = f^{(0,0,0,1,0)}[x[0], x[1], x[2], x[3], \psi]$

Corresponding global transformations:

X[1] gives $\{x[0][\epsilon] \to x[0], x[1][\epsilon] \to x[1], x[2][\epsilon] \to x[2] \operatorname{Cos}[\epsilon] - x[3] \operatorname{Sin}[\epsilon], x[3][\epsilon] \to x[3] \operatorname{Cos}[\epsilon] + x[2] \operatorname{Sin}[\epsilon], \psi[\epsilon] \to \psi\}$

X[2] gives $\{x[0][\epsilon] \to x[0], x[1][\epsilon] \to x[1] \operatorname{Cos}[\epsilon] + x[3] \operatorname{Sin}[\epsilon], x[3][\epsilon] \to x[3] \operatorname{Cos}[\epsilon] - x[1] \operatorname{Sin}[\epsilon], x[2][\epsilon] \to x[2], \psi[\epsilon] \to \psi\}$

X[3] gives $\{x[0][\epsilon] \to x[0], x[1][\epsilon] \to x[1] \operatorname{Cos}[\epsilon] - x[2] \operatorname{Sin}[\epsilon], x[2][\epsilon] \to x[2] \operatorname{Cos}[\epsilon] + x[1] \operatorname{Sin}[\epsilon], x[3][\epsilon] \to x[3], \psi[\epsilon] \to \psi\}$

X[4] gives $\left\{x[0][\epsilon] \to \frac{1}{2} e^{-\epsilon} \left(x[1] \left(-1 + e^{2\epsilon}\right) + x[0] \left(1 + e^{2\epsilon}\right)\right), x[1][\epsilon] \to \frac{1}{2} e^{-\epsilon} \left(x[0] \left(-1 + e^{2\epsilon}\right) + x[1] \left(1 + e^{2\epsilon}\right)\right), x[2][\epsilon] \to x[2], x[3][\epsilon] \to x[3], \psi[\epsilon] \to \psi\right\}$

X[5] gives $\left\{x[0][\epsilon] \to \frac{1}{2} e^{-\epsilon} \left(x[2] \left(-1 + e^{2\epsilon}\right) + x[0] \left(1 + e^{2\epsilon}\right)\right), x[2][\epsilon] \to \frac{1}{2} e^{-\epsilon} \left(x[0] \left(-1 + e^{2\epsilon}\right) + x[2] \left(1 + e^{2\epsilon}\right)\right), x[1][\epsilon] \to x[1], x[3][\epsilon] \to x[3], \psi[\epsilon] \to \psi\right\}$

X[6] gives $\left\{x[0][\epsilon] \to \frac{1}{2} e^{-\epsilon} \left(x[3] \left(-1 + e^{2\epsilon}\right) + x[0] \left(1 + e^{2\epsilon}\right)\right), x[3][\epsilon] \to \frac{1}{2} e^{-\epsilon} \left(x[0] \left(-1 + e^{2\epsilon}\right) + x[3] \left(1 + e^{2\epsilon}\right)\right), x[1][\epsilon] \to x[1], x[2][\epsilon] \to x[2], \psi[\epsilon] \to \psi\right\}$

X[7] gives $\{x[0][\epsilon] \to x[0] + \epsilon, x[1][\epsilon] \to x[1], x[2][\epsilon] \to x[2], x[3][\epsilon] \to x[3], \psi[\epsilon] \to \psi\}$

X[8] gives $\{x[0][\epsilon] \to x[0], x[1][\epsilon] \to x[1] + \epsilon, x[2][\epsilon] \to x[2], x[3][\epsilon] \to x[3], \psi[\epsilon] \to \psi\}$

X[9] gives $\{x[0][\epsilon] \to x[0], x[1][\epsilon] \to x[1], x[2][\epsilon] \to x[2] + \epsilon, x[3][\epsilon] \to x[3], \psi[\epsilon] \to \psi\}$

X[10] gives $\{x[0][\epsilon] \to x[0], x[1][\epsilon] \to x[1], x[2][\epsilon] \to x[2], x[3][\epsilon] \to x[3] + \epsilon, \psi[\epsilon] \to \psi\}$

Commutator table:

	1	2	3	4	5	6	7	8	9	10
1	0	-X[3]	X[2]	0	-X[6]	X[5]	0	0	-X[10]	X[9
2	X[3]	0	-X[1]	X[6]	0	-X[4]	0	X[10]	0	-X[
3	-X[2]	X[1]	0	-X[5]	X[4]	0	0	-X[9]	X[8]	0
4	0	-X[6]	X[5]	0	X[3]	-X[2]	-X[8]	-X[7]	0	0
5	X[6]	0	-X[4]	-X[3]	0	X[1]	-X[9]	0	-X[7]	0
6	-X[5]	X[4]	0	X[2]	-X[1]	0	-X[10]	0	0	-X[
7	0	0	0	X[8]	X[9]	X[10]	0	0	0	0
8	0	-X[10]	X[9]	X[7]	0	0	0	0	0	0
9	X[10]	0	-X[8]	0	X[7]	0	0	0	0	0
10	-X[9]	X[8]	0	0	0	X[7]	0	0	0	0