Change of variables in diferential equations

Run TMF064.Package.m first!

```
ChangeOfVariables[{D[x[t], t, t] + ω² x[t] == 0},
 {t → T + β, x[t] → α X[T]},
 {t}, {x},
 {T}, {X},
 showAll = 1
]
```

The highest order of derivative is 2

Jacobian (matrix A) is (1) and its inverse is (1)

Derivative of x with respect to {t}:

$\alpha X'[T]$

Derivative of x with respect to {t, t}:

$\alpha X''[T]$

$\{\alpha (\omega^2 X[T] + X''[T]) == 0\}$

Example of ordinary differential equation invariant under rotation

```
ChangeOfVariables[{(x - y[x]) D[y[x], x] == (x + y[x])},
 {x → u Cos[ϕ] + v[u] Sin[ϕ], y[x] → -u Sin[ϕ] + v[u] Cos[ϕ]},
 {x}, {y},
 {u}, {v},
 showAll = 1
]
```

The highest order of derivative is 1

Jacobian (matrix A) is (Cos[ϕ] + Sin[ϕ] v′[u]) and its inverse is $\left(\frac{1}{\mathrm{Cos}[\phi]+\mathrm{Sin}[\phi]\, v'[u]}\right)$

Derivative of y with respect to {x}:

$$\frac{-\mathrm{Sin}[\phi] + \mathrm{Cos}[\phi]\, v'[u]}{\mathrm{Cos}[\phi] + \mathrm{Sin}[\phi]\, v'[u]}$$

$$\left\{\frac{u + v[u] - u\, v'[u] + v[u]\, v'[u]}{\mathrm{Cos}[\phi] + \mathrm{Sin}[\phi]\, v'[u]} == 0\right\}$$

```
ChangeOfVariables[{(x - y[x]) D[y[x], x] == (x + y[x])},
 {x → r[ϕ] Cos[ϕ], y[x] → r[ϕ] Sin[ϕ]},
 {x}, {y},
 {ϕ}, {r},
 showAll = 1
]
```

```
The highest order of derivative is 1
```

Jacobian (matrix A) is $\left(-r[\phi]\,\text{Sin}[\phi] + \text{Cos}[\phi]\,r'[\phi] \right)$ and its inverse is $\left(\frac{1}{-r[\phi]\,\text{Sin}[\phi]+\text{Cos}[\phi]\,r'[\phi]} \right)$

Derivative of y with respect to {x}:

$$\frac{\text{Cos}[\phi]\,r[\phi] + \text{Sin}[\phi]\,r'[\phi]}{-r[\phi]\,\text{Sin}[\phi] + \text{Cos}[\phi]\,r'[\phi]}$$

$$\left\{\frac{r[\phi]\,(r[\phi] - r'[\phi])}{r[\phi]\,\text{Sin}[\phi] - \text{Cos}[\phi]\,r'[\phi]} == 0\right\}$$

Example of ordinary differential equation invariant under Galilean transformation

```
(* in general D[y[x],x,x]==f[x, x D[y[x],x]-y[x]] *)
ChangeOfVariables[{D[y[x], x, x] == f[x, x D[y[x], x] - y[x]]},
 {x → X, y[x] → Y[X] + ϵ X},
 {x}, {y},
 {X}, {Y},
 showAll = 1
]
```

```
The highest order of derivative is 2
```

Jacobian (matrix A) is (1) and its inverse is (1)

Derivative of y with respect to {x}:

$$\epsilon + Y'[X]$$

Derivative of y with respect to {x, x}:

$$Y''[X]$$

$$\{f[X, -Y[X] + X\,Y'[X]] == Y''[X]\}$$

Heat equation

```
ChangeOfVariables[{D[u[x, t], x, x] == D[u[x, t], t]},
 {x → X - 2 T ϵ, t → T, u[x, t] → U[X, T] Exp[X ϵ - T ϵ^2]},
 {x, t}, {u},
 {X, T}, {U},
 showAll = 1
]
```

```
The highest order of derivative is 2
```

Jacobian (matrix A) is $\begin{pmatrix} 1 & 0 \\ -2\in & 1 \end{pmatrix}$ and its inverse is $\begin{pmatrix} 1 & 0 \\ 2\in & 1 \end{pmatrix}$

Derivative of u with respect to {x}:

$$e^{\in (X - T\in)} \left(\in U[X, T] + U^{(1,0)}[X, T]\right)$$

Derivative of u with respect to {t}:

$$e^{\in (X - T\in)} \left(\in^2 U[X, T] + U^{(0,1)}[X, T] + 2\in U^{(1,0)}[X, T]\right)$$

Derivative of u with respect to {t, x}:

$$e^{\in (X - T\in)} \left(\in^3 U[X, T] + \in U^{(0,1)}[X, T] + 3\in^2 U^{(1,0)}[X, T] + U^{(1,1)}[X, T] + 2\in U^{(2,0)}[X, T]\right)$$

Derivative of u with respect to {t, t}:

$$e^{\in (X - T\in)} \left(\in^4 U[X, T] + 2\in^2 U^{(0,1)}[X, T] + U^{(0,2)}[X, T] + 4\in^3 U^{(1,0)}[X, T] + 4\in U^{(1,1)}[X, T] + 4\in^2 U^{(2,0)}[X, T]\right)$$

Derivative of u with respect to {x, x}:

$$e^{\in (X - T\in)} \left(\in^2 U[X, T] + 2\in U^{(1,0)}[X, T] + U^{(2,0)}[X, T]\right)$$

Derivative of u with respect to {x, t}:

$$e^{\in (X - T\in)} \left(\in^3 U[X, T] + \in U^{(0,1)}[X, T] + 3\in^2 U^{(1,0)}[X, T] + U^{(1,1)}[X, T] + 2\in U^{(2,0)}[X, T]\right)$$

$$\left\{e^{\in (X - T\in)} \left(U^{(0,1)}[X, T] - U^{(2,0)}[X, T]\right) == 0\right\}$$

3-dimensional Schrödinger equation

```
SchEq = Expand[ChangeOfVariables[
    {-1/2 (D[ψ[x, y, z], x, x] + D[ψ[x, y, z], y, y] + D[ψ[x, y, z], z, z]) +
        V[Sqrt[x^2 + y^2 + z^2]] ψ[x, y, z] - Energy ψ[x, y, z] == 0},
    {x → r Sin[θ] Cos[ϕ], y → r Sin[θ] Sin[ϕ], z → r Cos[θ],
     ψ[x, y, z]  → χ[r, θ, ϕ] / r},
    {x, y, z}, {ψ},
    {r, θ, ϕ}, {χ},
    showAll = 0
  ]]
```

$$\left\{2\,\text{Energy}\, r\,\chi[r, \theta, \phi] - 2\, r\, V\left[\sqrt{r^2}\right] \chi[r, \theta, \phi] + \frac{\text{Csc}[\theta]^2 \chi^{(0,0,2)}[r, \theta, \phi]}{r} + \frac{\text{Cot}[\theta]\, \chi^{(0,1,0)}[r, \theta, \phi]}{r} + \frac{\chi^{(0,2,0)}[r, \theta, \phi]}{r} + r\,\chi^{(2,0,0)}[r, \theta, \phi] == 0\right\}$$

```
Simplify[SchEq, Assumptions → r > 0]
```

$$\left\{2\, r^2 \left(\text{Energy} - V[r]\right) \chi[r, \theta, \phi] + \text{Csc}[\theta]^2 \chi^{(0,0,2)}[r, \theta, \phi] + \text{Cot}[\theta]\, \chi^{(0,1,0)}[r, \theta, \phi] + \chi^{(0,2,0)}[r, \theta, \phi] + r^2 \chi^{(2,0,0)}[r, \theta, \phi] == 0\right\}$$

- **Black–Scholes PDE (from Wikipedia - Change of variables (PDE)) transforms to the heat equation**

```
D[V[S, t], t] + S^2 D[V[S, t], S, S] / 2 + S D[V[S, t], S] - V[S, t] == 0
```

$$-V[S, t] + V^{(0,1)}[S, t] + S\, V^{(1,0)}[S, t] + \frac{1}{2} S^2 V^{(2,0)}[S, t] == 0$$

```
ChangeOfVariables[
 {D[V[S, t], t] + S^2 D[V[S, t], S, S] / 2 + S D[V[S, t], S] - V[S, t] == 0},
 {S → Exp[x], t → T - 2 τ, V[S, t] → Exp[-x / 2 - 9 τ / 4] u[x, τ]},
 {S, t}, {V},
 {x, τ}, {u},
 showAll = 0
]
```

$$\left\{e^{-\frac{x}{2}-\frac{9\tau}{4}} \left(u^{(0,1)}[x, \tau] - u^{(2,0)}[x, \tau]\right) == 0\right\}$$

- **Hamiltonian of two particles in 1D expressed in the center-of-mass system**

```
ChangeOfVariables[
 {-D[ψ[X1, X2], X1, X1] / (2 * m1) - D[ψ[X1, X2], X2, X2] / (2 * m2) == 0},
 {X1 → XT + m2 * XR / (m1 + m2), X2 → XT - m1 * XR / (m1 + m2), ψ[X1, X2] → Ψ[XT, XR]},
 {X1, X2}, {ψ},
 {XT, XR}, {Ψ},
 showAll = 0
]
```

$$\left\{\left(\left(m1 + m2\right)^2 \Psi^{(0,2)}[XT, XR] + m1\, m2\, \Psi^{(2,0)}[XT, XR]\right) \Big/ \left(m1\, m2\, \left(m1 + m2\right)\right) == 0\right\}$$